Mental constructions for the learning of the concept of vector space

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We present the results of the implementation of a teaching strategy for the learning of the concept of vector space. The strategy was implemented with a group of engineering students and its design is based on the ACE methodology from APOS theory. The results show that working with sets and binary operations different from those traditionally handled in a first course of linear algebra, promotes students' reflection on the validity of the axioms that define vector spaces and on the properties of the zero vector and the additive inverses.

Keywords: vector space, linear algebra learning, APOS theory

INTRODUCTION

Many of the particular obstacles that arise in the linear algebra teaching and learning processes are related to the nature of the elements that constitute this mathematical theory. Linear algebra is formed by a network of interconnected definitions, axioms, and abstract theorems. This results in frequent difficulties among students to succeed in higher level courses related to linear algebra. Thus, students end up confused and disoriented when trying to understand concepts related to this discipline, such as vector spaces, subspaces, linear transformations, among others (Dorier & Sierpinska, 2001).

This paper reports on the development and implementation of a teaching strategy to promote the understanding of the vector space concept among engineering students. Its design is informed by APOS theory. Due to the COVID-19 pandemic this teaching strategy was implemented online. The participating students had not received training in mathematical logic, nor did they have experience working with argumentative or demonstrative processes in their previous mathematics courses.

RESEARCH QUESTION AND OBJECTIVES

This research study is part of a wider project which aims to analyze the mental constructions evidenced by a group of students when working with the concept of vector space. Students had followed a linear algebra course where activities designed with APOS theory (Actions, Processes, Objects, Schemas) were used, and the APOS teaching methodology, involving Activities, Class discussions and Exercises (ACE teaching cycle), was followed. The study aims to address in general the following research question:

What mental constructions related to vector space do students manifest after finishing a course designed with APOS theory?

The teaching strategy used throughout the course included work on activities aimed at learning concepts that have been identified previously (Parraguez & Oktaç, 2010) as a requirement for the construction of the concept of vector space. This led us to consider another related research question:

What is the impact of introducing the study of the concepts of equality, set and binary operation, as background, in a teaching strategy aimed at supporting the construction of the concept of vector space?

The research Project conducted resulted in information regarding both questions. However, we decided to focus in this paper on results obtained from the data analysis corresponding to the second one. Space restrictions did not allow to include a detailed report on results obtained throughout the whole research experience.

LITERATURE REVIEW

In order to identify the mathematical concepts required for the learning of the concept of vector space, we conducted a literature review focusing on the main learning obstacles related to this concept (Can et al., 2021).

A first epistemological obstacle that students face comes from the level of formalism inherent to linear algebra (Dorier, 1998). In particular, when facing the concept of vector space, students may face many of the obstacles associated with the formalism and level of abstraction needed in its study. For instance, when working with the demonstrative and argumentative processes that are required to verify whether or not a given set is a vector space (Mutambara & Bansilal, 2018); or when working with the concept of zero vector and the additive inverse vectors of a space where elements are not necessarily n-tuples. These obstacles are also encountered when working with vector spaces where addition and scalar multiplication are defined differently from those traditionally defined in $\mathbb{R}^n$ (Kú et al., 2008; Parraguez & Oktaç, 2010).

Many students have difficulty recognizing the characteristics defining both the zero vector and the additive inverses. Some authors have suggested the use of unusual vector spaces as a way to address this situation, for example, vector spaces where the pre-established algorithms that traditionally work on typical vector spaces are not sufficient. In order to succeed, flexibility is necessary to identify those characteristics shared by the sets and the binary operations that constitute such vector spaces (Parraguez & Oktaç, 2012; Parraguez, 2013).

It is important to underline that, in spite of reports in the literature on the need to take into account the construction of the notion of set and binary operation before introducing vector space, it was difficult to find in the literature studies that take them into consideration as part of their research objective. We posit this may be due to researchers and teachers considering these concepts as something known by students, although this may not be the case for students taking a first course on Linear Algebra at the university, particularly if they are enrolled in fields other than mathematics.

The findings of this literature review put forward the need to consider the introduction of the concepts of set, binary operation, axiom, and function as requirements before dealing explicitly with the construction of the concept of vector space. Thus, in this paper we present a teaching strategy that considers all these elements.

The obstacles described above together with teaching suggestions given by some authors served us as the basis for designing a genetic decomposition addressing specifically the construction of the zero vector and additive inverses. They have also helped us to design a teaching strategy that introduces students to working with vector spaces whose elements are not necessarily n-tuples.

THEORETICAL FRAMEWORK

This research is based on APOS theory (Arnon et al., 2014). This theory intends to understand the constructions students need to learn a concept. Its main conceptual structures follow:

Actions are defined as the transformations applied to previously constructed Objects and that are somehow external to the subject that applies them. They are identified by the fact that the subject needs an external stimulus to apply them.

Processes are understood as the Actions interiorized by the subject, in such a way that the individual is capable of reflecting on such Actions without the need for external stimuli, and can describe them or even reverse the steps without the need to perform the steps operationally. Processes can be coordinated into other Processes and can also be reversed.

When an individual reflects on the operations applied to a particular Process, becomes aware of the Process as a whole and is able to perform new Actions on it, that is, can act on the Process itself, the individual has encapsulated the Process into an *Object*. Furthermore, if the subject is able to go back from the Object to the Process from where it comes, it could be said that the individual has de-encapsulated the Object into a Process.

A *Schema* is constructed as a coherent collection of mental structures (Actions, Processes, Objects and other Schemas) and the relationships between them. A Schema is described in terms of the mental structures that compose it and how they are related to each other. Schemas are constantly developing through a triad of stages: Intra- , Inter -, and Trans- . When a Schema has been constructed at the Trans- stage and can be considered as coherent, it can be thematized into an Object and new Actions can be applied to it.

METHODOLOGY

APOS theory research methodology begins with an analysis that leads to the development of a model for the epistemology of the concept to be studied. This model—which is called the *genetic decomposition* of the concept—is based on the description of the Actions, Processes, Objects and Schemas and the corresponding mechanisms needed in the construction of the concept or concepts of interest (Arnon et al., 2014). This model is not intended to be unique and needs to be validated through research. We describe the genetic decomposition below.

The genetic decomposition was used to develop a set of twenty tasks for the introduction of vector space. These tasks were organized in six activity sets to be used in the classroom. The design of the tasks included the construction of the concepts of equality, set, and binary operation—in addition to those related to the construction of the concept of vector space. As mentioned before, these elements were identified as a necessary background for learning vector spaces (e.g., Parraguez & Oktaç, 2010).

The teaching strategy followed the ACE cycle. The experience was conducted with a group of 20 engineering students enrolled in a first linear algebra course at a public university. It is important to underline that these students had not been introduced before neither to these concepts not to work with operations different to those defined traditionally for sets other than $\mathbb{R}^n$. These were the reasons to design tasks including different sets and operations. Also we considered that working with them would foster students' reflection on the need of the axioms defining vector spaces.

Students were organized in teams of four to work in the solution of the designed activity sets through six sessions. Due to the quarantine imposed by COVID-19, all the sessions took place on line.

The teaching strategy implemented in all sessions was the ACE Cycle: it consisted in students working collaboratively in small groups on the activities (A). Collaborative work was followed by whole class discussion with the teacher (C). These two steps were repeated several times during each session and homework exercises (E) were handed to students at the end of each class.

The first three sessions were devoted to work on the construction of the pre-requisite concepts, namely equality, sets and binary operations through tasks involving proving some vector space axioms related to them. The last three sessions consisted in defining of vector space followed by tasks involving different sets and binary operations where students had to prove if they were vector spaces. Students worked on these last tasks using paper and pencil and then programming computer codes to construct the vector space axioms.

All the work produced by the students during the sessions was kept and the video recording and work during interviews of one student in each group at the end of the semester was analyzed by the three researchers and results were negotiated. Analysis focused on describing and identifying important emerging ideas during small groups work, the evolution of students' contributions during whole group discussion and evidence of the construction and development of APOS structures related to the pre-requisite concepts and the vector space throughout the sessions and the interview for each student.

In this document we show results from three students that exemplify the role played by the use of tasks designed with the genetic decomposition and the construction of pre-requisite concepts on students' construction of the vector space concept.

Genetic decomposition of the vector space concept.

Genetic decompositions of the prerequisite concepts were designed taking into account results from the literature. The first three sets of activities implemented were devoted to students' construction of those prerequisites.

We proposed a new genetic decomposition for the vector space concept. It combines elements from the decompositions proposed by Parraguez and Oktaç (2010) and Arnon et al. (2014). This new genetic decomposition considers as prerequisites mental constructions of the concepts of equality, set, and binary operation. Next, this new genetic decomposition is introduced. It includes a mechanism to integrate the axiom schema in the construction of the vector space concept. The role of logical quantifiers of existence and universality to characterize the axioms that define a vector space is also considered.

The construction of the concept of vector space is based on the construction of the Schemas of the concepts of set, binary operation and equality: the student begins with the Action of calculating the result of applying a given binary operation to specific elements of the same set, and to the Cartesian products of sets. With these Actions, binary operations are conceived as functions with certain input arguments that in turn produce an output.

By working on the application of different binary operations on all the elements of different sets to analyze and describe their Actions, without having to carry out the operations explicitly, the interiorization of the set Process is promoted. This Process is defined in terms of a membership condition and the binary operation Process defined on a set that can be coordinated into a new one to originate a Process of the notion of sets with binary operations.

Continuing with the construction of the concept of vector space, the concept of axiom is constructed as follows: given a property of a binary operation involving the equality sign, the individual applies the Action of evaluating both members of equality on specific elements. This is done based on the definition of the binary operation to determine the validity of the equality stated by the property.

In order to continue with the construction of the vector space concept, the concept of axiom needs to be constructed as follows: Given a binary operation property involving the equal sign, the student does the Action of evaluating it in specific elements appearing in the two sides of the equality in terms of the definition of the binary operation concerned and then determines the validity of the given equality in terms of the specified property, When the individual reflects on the validity of a property of binary operations defined on a set, applied to all the elements of the set—involving the universal quantifier—, she interiorizes such Actions into a verification Process of an axiom with a universal quantifier. This Process, in turn, is coordinated with the Process set with binary operation to give rise to the verification Process of an axiom with universal quantifier for binary operations on sets.

This implies that the individual can analyze and discuss the existence of a specific element that satisfies an axiom, without the need to list all the elements of the set, and that she is able to determine the missing element in the expression that involves the axiom. This Process is coordinated with the sets with binary operations Process into the verification Process of an axiom with existential quantifier for binary operations on sets.

The two previous axiom- verification Processes are coordinated with the Processes corresponding to the axioms that define the vector space. This results in a set with binary operations Process (addition and scalar multiplication) that satisfy the axioms.

The resulting structure is a dynamic structure that depends on the set and the binary operations that are considered. This structure becomes static when the individual conceives the vector space as an object to which she can apply Actions such as finding a base, finding a spanning set, determining the linear dependence or independence of a set of elements interpreted as vectors, and applying linear transformations between vector spaces.

RESULTS AND DISCUSSION

As mentioned before, after presenting the definition of vector space students worked on testing the axioms and deciding if the given set was or not a vector space. We present now an analysis of transcripts form the interventions of three students when they addressed one of the tasks worked in session four posed to the students with the intention of observing the constructions that they evidenced when working with the concept of vector space.

Task.

Verify if the set $V = \left\{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : x, y, z \in \mathbb{R}^{+}\right\}$ is a vector space with the operations defined below.

Vector addition: $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \oplus \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & be \\ 0 & cf \end{pmatrix}$.

Scalar multiplication: $t \otimes \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a^t & b^t \\ 0 & c^t \end{pmatrix}$.

When questioned about the algebraic closure of the defined addition, student 1 answered as follows:

Student 1: Yes, it does comply because... well... it does comply because in both matrices the elements that are taken, well, as it says there, as a rule, they belong to the positive real numbers and when doing the sum, well ... it will b the same. The sum of these matrices remains the same, so to speak, with the elements belonging to V.

Interviewer: Explain to me, what do you mean when you say that it stays the same?

Student 1: Yes... it could be, for example, ... a matrix of 1, 2, 0, 3... we do the sum of a matrix of 2, 4, 0, 6, and we would obtain a matrix of 2, 8, 0, 18.

A second student makes the following comments:

Student 2: It is like this... *ad*, *be*, 0 and *c* multiplied by *f*, but it was also replaced by numbers, like this with numbers, to verify that they belong to the positive real numbers as it is in the vector and they do belong… um, I wrote that... the matrix one is 1, 2, 0 and 3 and the other one is 4, 5, 0 and 6 and well... I replaced the numbers with the letters, and at the end I obtained 4, 8, 0 and 18.

Interviewer: Okay, and how does the calculation explain that the property is true?

Student 2: Because they belong to the set V, of positive real numbers.

To verify the closure property of vector addition, both students select specific vectors from the given sets and calculate the sum with those elements. This illustrates one of the first Actions most of the students perform when they have to verify that a set is a vector space. They select specific values to verify that the axioms are satisfied and, in some instances, they could conclude—based on particular cases—that such axioms are satisfied by all the elements in the set, without analyzing the validity of their conclusion on all the elements of the set. After analyzing the rest of student' interventions, we concluded that twelve students showed similar interventions corresponding to the construction of Actions.

When questioning a third student regarding the validity of the closure of the addition using the same task, he replies:

Student 3: Well they were like... It was a set like this... in the first row *a*, *b*, *c*, in the second row 0, *c*, and the other element would be, the first row *d*, *e* and the second row 0, *f*, so those two elements. That is, we apply the sum defined for the set... And, well, we got the matrix, so to speak, in the first line *ad*, *be*, 0 and *c* multiplied by *f*.

Interviewer: How did you get that?

Student 3: Because we applied the operation that was indicated for the set... from what we understood, it was like that. Because it says that *u*, the... the sum of vectors with *v* belongs to *V*. It means that if you take an element and add it with an element of *V* and... it means that these two elements are in *V* ... I don't know if you understand me... That is, the result is the one that has to be in *V*, right? because as I had already said before... The axiom tells us that *u* and *v* are elements that are in *V*... then obviously they comply with the... with the shape and structure of the set *V*. It means that, if when applying the operation to those two elements, yes... that axiom... for that axiom to hold true, the result of that sum must also belong to *V*... That's more or less what it means, right?

Student 3 argues about the validity of the closure axiom for the addition of vectors, using the definition of the set and the operation. He shows a generalization of the verification of the closure on different elements. This is evidence of a Process construction related to the verification of the axioms.

When questioned about the existence of additive inverse vectors, student 3 comments (see Figure 1):

Student 3: But if you notice… you are dividing *a* by *a*, well obviously if you have... let's say that *a* is equal to 3, 3 divided by 3 will give you 1. So if they realize, *a* over *a*, is practically 1 and, the others are 1 all those who are there...

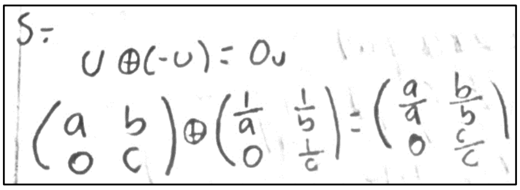

Figure1: Vector proposed by student 3 to verify the existence of additive inverses.

Interviewer: Well, there we have the image, so in the image we see a matrix *u* that is *a*, *b*, 0 and *c*, and it is being added to another matrix that is 1 over *a*, 1 over *b*, 0 and 1 over *c*... Can you explain that?

Student 3: Well, I took as a reference the one we already had in the fourth [axiom], since it is indicating that, if -*u* exists, it should lead us to the value of the additive neutral element. And well, since we had already stated that the additive neutral element... well, I put it as I had represented it before, then I used the 1 over *a*, since when doing the multiplication, I mean, when doing the addition operation, the multiplication would be done linearly and it would be like *a* over *a*, and it would give us as a result the additive neutral element that we had already obtained before.

Student 3 proposes the general structure of the additive inverse vector based on a generic vector *u* of the given set, and applies the binary operation defined on this set. In his intervention, student 3 describes the way in which the structure he proposes for the inverse vector can serve to prove the existence of the additive inverse elements for a specific value. This evidences a Process conception of verification of axioms with existential quantifier for binary operations defined on sets. After analyzing the rest of students' interventions, we found eight students showing evidence of a Process conception of vector space.

Finally, we found that only one student showed evidence of the construction of vector space as an Object.

In general, the mental structures that were introduced in the genetic decomposition of the vector space concept to address the verification of axioms with universal and existential quantifiers, were evidenced during the students' discussions that emerged when looking for specific elements that satisfied those axioms. This discussion involved reflection on binary operations and definitions of sets, which confirms the importance of these concepts for the construction of vector spaces as proposed by Parraguez and Oktaç (2010).

The coordination of the Process related to applying a binary operation to all the elements of a set, and the Processes that involve evaluating axioms with quantifier of existence and uniqueness, was evidenced in the students' discussion about the characteristics of the neutral element and the additive inverses. This coordination involved reflections on the particular characteristics of the elements that constitute a set and the way in which binary operations act on all these elements. This led the students to reflect on the uniqueness and universality of the neutral element and the additive inverses.

This situation paved the way for the verification of the particular characteristics of these vectors in a natural way within the construction of the concept of vector space, and is related to the Process of verification of an axiom with existential quantifier and the Process of verification of an axiom with universal quantifier for binary operations.

CONCLUSIONS

Working with vector spaces requires managing different sets of vectors, applying different types of binary operations and verifying that the axioms that define vector spaces are satisfied by the given elements of a vector space (Kú et al., 2008; Parraguez & Oktaç, 2012; Parraguez, 2013). The research reported in this paper has addressed these elements through a genetic decomposition proposed for the construction of the concept of vector space that includes the background that is considered necessary for learning this concept.

It was observed that working with different sets and analyzing their elements based on their definition, favors the ability to describe and propose the vectors that belong to a particular vector space. This allow students to recognize which elements are part of a set based on its definition, and propose specific elements that can be evaluated in binary operations.

The use of different binary operations turned out to be a strategy that favors the interiorization of the Actions carried out by the individual when evaluating the operation on particular elements. In particular, the work developed with different types of binary operations favors the ability to describe the way in which the operations act on all the elements of a given set. An observed phenomenon in this sense involves the use of literals to describe the way in which a given operation acts: when the student uses them to operate on a generic element of a set and describe the way in which an operation acts on all the elements of the set, she evidences the construction of a Process conception.

The didactic strategy based on collaborative work between students and with the teacher together with activities designed with the genetic decomposition proved to be effective in fostering students' reflection and understanding of the role of axions and their need in the definition of vector space.

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